

A VINOGRADOV-TYPE PROBLEM IN ALMOST PRIMES

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ABSTRACT. We prove a generalisation of Vinogradov's theorem by finding for $m \geq 3$ and fixed positive integers $c_1, \dots, c_m, r_1, \dots, r_m$ the asymptotics of the number of sequences $(n_1, \dots, n_m) \in \mathbf{N}^m$ such that $c_1 n_1 + \dots + c_m n_m = N$ and $\Omega(n_i) = r_i$ for every $i = 1, \dots, m$ under the assumption that at least three of the r_i are equal to 1.

1. INTRODUCTION

One of the most famous problems of additive combinatorics which are already solved is the so called weak (or ternary) Goldbach conjecture, which can be stated in the following form:

Theorem 1.1. *Every odd number N greater than 1 is a sum of at most three primes.*

The assertion of Theorem 1.1 was proven to be correct for all sufficiently large N in 1937 by Vinogradov [1]. Later Chen and Wang [2] gave an effective proof i.e. for all $N \geq \exp(\exp(11.503))$. This threshold was lowered to $N \geq \exp(3100)$ by Liu and Wang [3], but it was still too weak to prove ternary Goldbach conjecture for all lower positive integers using computer calculations¹. In 2012 and 2013 Helfgott in [4], [5] gave new bounds which were strong enough to verify all remaining cases directly.

The proof of the ineffective version of Theorem 1.1 gave us also the precise asymptotic of the number of solutions of the equation $p_1 + p_2 + p_3 = N$ for p_1, p_2, p_3 being primes where N is a variable. For technical reasons it is easier to attach a weight $\log p$ to each power of a prime p and deal with the sum

$$R_3(N) = \sum_{\substack{n_1, n_2, n_3 \\ n_1 + n_2 + n_3 = N}} \Lambda(n_1) \Lambda(n_2) \Lambda(n_3),$$

where $\Lambda(n)$ denotes the von Mangoldt function.

In this paper we calculate the asymptotics of the number of solutions in almost primes of the more general equation $c_1 n_1 + \dots + c_m n_m = N$, where the c_i are some fixed positive integers. Formally, we assume that the number $\Omega(n_i)$ of prime divisors of n_i is equal to r_i , where (r_1, \dots, r_m) is a sequence of positive integers independent of N . There are obstacles which force us to assume that at least three of the r_i are equal to 1. By using the Hardy-Littlewood circle method and some combinatorial arguments we are able to prove the following result

Theorem 1.2. *Fix $m \geq 3$. Let c_1, \dots, c_m be some positive integers satisfying $(c_1, \dots, c_m) = 1$, and let r_1, \dots, r_m be a sequence of positive integers which contains at least three elements equal to 1. Then for $N > 20$*

¹However, a complete proof was presented in [6] on the assumption that the generalised Riemann hypothesis is true.

$$\sum_{\substack{n_1, \dots, n_m \\ c_1 n_1 + \dots + c_m n_m = N \\ \Omega(n_1) = r_1, \dots, \Omega(n_m) = r_m}} 1 = \frac{1}{(m-1)!} \frac{1}{(r_1-1)! \dots (r_m-1)!} \frac{1}{c_1 \dots c_m} \times$$

$$\frac{N^{m-1}}{\log^m N} (\log \log N)^{r_1 + \dots + r_m - m} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)),$$

where

$$\mathfrak{S}_{c_1, \dots, c_m}(N) = \prod_{p|N} \left(1 + \frac{\mu\left(\frac{p}{(c_1, p)}\right) \dots \mu\left(\frac{p}{(c_m, p)}\right)}{\varphi\left(\frac{p}{(c_1, p)}\right) \dots \varphi\left(\frac{p}{(c_m, p)}\right)} (p-1) \right) \prod_{p \nmid N} \left(1 - \frac{\mu\left(\frac{p}{(c_1, p)}\right) \dots \mu\left(\frac{p}{(c_m, p)}\right)}{\varphi\left(\frac{p}{(c_1, p)}\right) \dots \varphi\left(\frac{p}{(c_m, p)}\right)} \right).$$

The first part of the proof is based on standard arguments developed by Vinogradov to calculate the number of solutions of $b_1 n_1 + \dots + b_m n_m = N$ in weighted primes, where we let the b_i depend somehow on N (precisely, we assume that $b_i \ll N^{1/12m}$). After that, we transmute the von Mangoldt weights into standard indicators and show how this affects the asymptotic. The final step (the most technically involved) is to use the definition of the b_i and some combinatorial arguments to get the desired asymptotics.

In big O or \ll notation the dependence on absolute constants will not be emphasized in any way.

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2. MAJOR ARCS

In the next two sections, we are going to find the asymptotics of the sum

$$\sum_{\substack{n_1, \dots, n_m \leq N \\ b_1 n_1 + \dots + b_m n_m = N}} \Lambda(n_1) \dots \Lambda(n_m)$$

with varying coefficients b_i via the circle method. Let $Q = \log^B N$ for some $B > 0$, which is going to be fixed later. Then for $q \leq Q$ and a , such that $(a, q) = 1$, we define a major arc in the usual manner:

$$\mathcal{M}_{a,q} := \left\{ \alpha \in \mathbf{R} : \left\| \alpha - \frac{a}{q} \right\|_{\mathbf{R}/\mathbf{Z}} \leq \frac{Q}{N} \right\}.$$

Let us also denote the sum of all major arcs by

$$\mathcal{M} := \bigcup_{q \leq Q} \bigcup_{\substack{a=1 \\ (a,q)=1}}^q \mathcal{M}_{a,q}.$$

Put $S(x, \alpha) = \sum_{n \leq x} \Lambda(n) e(n\alpha)$, $S_i(x, \alpha) = \sum_{n \leq \frac{x}{b_i}} \Lambda(n) e(nb_i \alpha)$ and $\mathfrak{m} = \mathbf{R}/\mathbf{Z} \setminus \mathcal{M}$.

Theorem 2.1. Fix $c_1, \dots, c_m \in \mathbf{N}$. Let $b_1, \dots, b_m \leq N^\delta$ for any $\delta \in (0, \frac{1}{12m})$ and let $b_i = c_i \eta_i$ for $i = 1, \dots, m$, where η_i is a positive integer with prime divisors greater than Q . Let us

further assume, that $(b_1, \dots, b_m) = 1$. Then for every $\varepsilon > 0$, we have

$$\sum_{\substack{n_1, \dots, n_m \leq N \\ b_1 n_1 + \dots + b_m n_m = N}} \Lambda(n_1) \dots \Lambda(n_m) = \frac{1}{(m-1)! b_1 \dots b_m} \mathfrak{S}_{c_1, \dots, c_m}(N) + \frac{1}{b_1 \dots b_m} O\left(\frac{N^{m-1}}{Q^{m-2-\varepsilon}}\right) + \int_{\mathfrak{m}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha.$$

Proof. Let us define $u(y) = \sum_{n \leq N} e(ny)$, $u_i(y) = \sum_{n \leq \frac{N}{b_i}} e(nb_i y)$ and put $\alpha = \frac{a}{q} + y$. Recall the well known identity (for proof see [9]) for $a, q \in \mathbf{N}$ coprime and $q < Q$:

$$(2.1) \quad S(N, \alpha) = \frac{\mu(q)}{\varphi(q)} u(y) + O\left((1 + N|y|) N \sqrt{q} \exp\left(-c\sqrt{\log N}\right)\right),$$

where c is some positive constant. From $S_i(N, \alpha) = S(N/b_i, b_i \alpha)$ and $b_i \leq N^{1/12m}$ we simply get

$$(2.2) \quad S_i(N, \alpha) = \frac{\mu\left(\frac{q}{(b_i, q)}\right)}{\varphi\left(\frac{q}{(b_i, q)}\right)} u_i(y) + O\left((1 + N|y|) \frac{N}{b_i} \sqrt{q} \exp\left(-c\sqrt{\log \frac{N}{b_i}}\right)\right).$$

Applying $1 \ll Q$, $|y| \leq \frac{Q}{N}$, $q \leq Q$ and $b_i < N^\delta$ we can estimate

$$(2.3) \quad (1 + N|y|) \frac{N}{b_i} \sqrt{q} \exp\left(-c\sqrt{\log \frac{N}{b_i}}\right) \ll \frac{NQ^{3/2}}{b_i} \exp\left(-c_1 \sqrt{\log N}\right),$$

where c_1 is some positive constant. Using $Q \ll \exp(\varepsilon \sqrt{\log N})$ for any $\varepsilon > 0$ we finally conclude

$$(2.4) \quad S_i(N, \alpha) = \frac{\mu\left(\frac{q}{(b_i, q)}\right)}{\varphi\left(\frac{q}{(b_i, q)}\right)} u_i(y) + O\left(\frac{N}{b_i} \exp\left(-C\sqrt{\log N}\right)\right)$$

for some positive constant C . Right now we are ready to estimate the contribution of a single major arc to the integral.

$$(2.5) \quad \int_{\mathcal{M}_{a, q}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha = \prod_{i=1}^m \frac{\mu\left(\frac{q}{(b_i, q)}\right)}{\varphi\left(\frac{q}{(b_i, q)}\right)} e\left(-\frac{aN}{q}\right) \times \int_{-\frac{Q}{N}}^{\frac{Q}{N}} \prod_{i=1}^m u_i(y) e(-Ny) dy + O\left(\sum_{\substack{\omega \in \{0,1\}^m \\ \omega \neq (1, \dots, 1)}} \int_{-\frac{Q}{N}}^{\frac{Q}{N}} f_{\omega_1}(y) \dots f_{\omega_m}(y) e(-Ny) dy\right),$$

where

$$(2.6) \quad f_{\omega_j}(y) := \begin{cases} u_i(y) \times \mu\left(\frac{q}{(b_i, q)}\right) / \varphi\left(\frac{q}{(b_i, q)}\right) & \text{if } \omega_j = 1 \\ \frac{N}{b_i} \exp\left(-C\sqrt{\log N}\right) & \text{if } \omega_j = 0 \end{cases}.$$

By the obvious inequality $|u_i(y)| \leq \frac{N}{b_i}$, if at least one coordinate of ω is equal to 0, then for such ω we get

$$(2.7) \quad \int_{-\frac{Q}{N}}^{\frac{Q}{N}} f_{\omega_1}(y) \dots f_{\omega_m}(y) e(-Ny) dy \ll \frac{1}{b_1 \dots b_m} \frac{N^{m-1}}{e^{(C-\varepsilon)\sqrt{\log N}}}.$$

Summing over every admissible a, q one gets

$$(2.8) \quad \int_{\mathcal{M}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha = \sum_{q \leq Q} \prod_{i=1}^m \frac{\mu\left(\frac{q}{(b_i, q)}\right)}{\varphi\left(\frac{q}{(b_i, q)}\right)} \sum_{\substack{a \leq q \\ (a, q)=1}} e\left(-\frac{aN}{q}\right) \times \int_{-\frac{Q}{N}}^{\frac{Q}{N}} \prod_{i=1}^m u_i(y) e(-Ny) dy +$$

$$\frac{1}{b_1 \dots b_m} O\left(\frac{N^{m-1}}{e^{(C-3\varepsilon)\sqrt{\log N}}}\right)$$

Put $c_q(N) = \sum_{\substack{a \leq q \\ (a, q)=1}} e\left(-\frac{aN}{q}\right)$. We can show that the sum over q on the right hand side of the equation (2.8) equals

$$(2.9) \quad \sum_{q \leq Q} \prod_{i=1}^m \frac{\mu\left(\frac{q}{(b_i, q)}\right)}{\varphi\left(\frac{q}{(b_i, q)}\right)} c_q(N) = \prod_{p|N} \left(1 + \prod_{i=1}^m \frac{\mu\left(\frac{p}{(c_i, p)}\right)}{\varphi\left(\frac{p}{(c_i, p)}\right)} (p-1)\right) \prod_{p \nmid N} \left(1 - \prod_{i=1}^m \frac{\mu\left(\frac{p}{(c_i, p)}\right)}{\varphi\left(\frac{p}{(c_i, p)}\right)}\right) +$$

$$O\left(\sum_{q > Q} \prod_{i=1}^m \frac{\mu\left(\frac{q}{(c_i, q)}\right)}{\varphi\left(\frac{q}{(c_i, q)}\right)} c_q(N)\right) = \mathfrak{S}_{c_1, \dots, c_m}(N) + O\left(\frac{1}{Q^{m-2-\varepsilon}}\right).$$

Moreover,

$$|\mathfrak{S}_{c_1, \dots, c_m}(N)| < \prod_p \left(1 + \frac{1}{\varphi(p)^{m-1}}\right) \ll 1.$$

Right now we only need to estimate the integral on the right hand side of (2.8). Let us consider the following subsets of \mathbf{R}/\mathbf{Z} :

$$J_k^{(j)} = \left[\frac{k}{b_j} - \frac{1}{b_j N^{1/3}}, \frac{k}{b_j} + \frac{1}{b_j N^{1/3}} \right],$$

$$I_k^{(j)} = \left[\frac{k}{b_j} - \frac{1}{b_j N^{1/2}}, \frac{k}{b_j} + \frac{1}{b_j N^{1/2}} \right],$$

for $j = 1, \dots, m$ and $k = 0, \dots, b_i - 1$. The distance between two fractions of the form k/b_j satisfies

$$(2.10) \quad \left| \frac{k_1}{b_{j_1}} - \frac{k_2}{b_{j_2}} \right| \geq \frac{1}{b_{j_1} b_{j_2}} \geq \max \left\{ \frac{1}{b_{j_1} N^\delta}, \frac{1}{b_{j_2} N^\delta} \right\} \geq \frac{1}{b_{j_1} N^{1/3}} + \frac{1}{b_{j_2} N^{1/3}}$$

for $N > 2^{\frac{3}{1-3\delta}}$ and arbitrary $k_1, k_2 \in \mathbf{Z}$. Consequently, we can assume that N is so large that every two intervals $J_k^{(j)}$ centered in different points k/b_j have empty intersections.

According to Appendix A we define $J_{b_1, \dots, b_m}(N)$ as a number of tuples of the form $(n_1, \dots, n_m) \in \mathbf{N}^m$ which fulfils $b_1 n_1 + \dots + b_m n_m = N$. From $I_k^{(j)} \subset J_k^{(j)}$ one can decompose

$$(2.11) \quad J_{b_1, \dots, b_m}(N) = \int_{\mathbf{R}/\mathbf{Z}} \prod_{i=1}^m u_i(y) e(-Ny) dy =: \int_{\mathbf{R}/\mathbf{Z}} = \int_{-\frac{Q}{N}}^{\frac{Q}{N}} + O\left(\sum_{j=1}^m \sum_{k=1}^{b_j-1} \left| \int_{I_k^{(j)}} + \int_{J_k^{(j)} \setminus I_k^{(j)}} \right| \right) + \int_{\frac{Q}{N}}^{\frac{1}{bN^{1/3}}} + \int_{-\frac{1}{bN^{1/3}}}^{-\frac{Q}{N}} + \int_{\mathcal{S}},$$

where $b := \min\{b_1, \dots, b_m\}$ and \mathcal{S} denotes the set $\mathbf{R}/\mathbf{Z} \setminus \bigcup_{j=1}^m \bigcup_{k=0}^{b_j-1} J_k^{(j)}$. From basic Dirichlet kernel estimations one gets

$$(2.12) \quad |u_i(y)| \ll \begin{cases} \frac{1}{N^{1/3}} & \text{if } y \notin J_k^{(i)} \\ \frac{1}{N^{1/2}} & \text{if } y \notin I_k^{(i)} \\ \frac{1}{|b_i y|} & \text{if } y \in [-\frac{1}{bN^{1/3}}, \frac{1}{bN^{1/3}}] \setminus \{0\} \\ N & \text{always} \end{cases},$$

(the second inequality from the bottom is true because for N sufficiently large we have $|b_i y| \leq \frac{b_i}{N^{1/3}} < \frac{1}{2}$, which gives $\|b_i y\|_{\mathbf{R}/\mathbf{Z}} = |b_i y|$) and thus

$$(2.13) \quad \left| \prod_{i=1}^m u_i(y) \right| \ll \begin{cases} N^{m-1+\frac{1}{3}} & \text{if } y \in \bigcup_{j=1}^m \bigcup_{k=1}^{b_j-1} I_k^{(j)} \\ N^{\frac{m-1}{2}+\frac{1}{3}} & \text{if } y \in \bigcup_{j=1}^m \bigcup_{k=1}^{b_j-1} J_k^{(j)} \setminus I_k^{(j)} \\ \frac{1}{b_1 \dots b_m} \frac{1}{|y|^m} & \text{if } y \in [-\frac{1}{bN^{1/3}}, \frac{1}{bN^{1/3}}] \setminus \{0\} \\ N^{\frac{m}{3}} & \text{if } y \in \mathcal{S} \end{cases}.$$

The two inequalities from the top follow from the fact that if $(b_1, \dots, b_m) = 1$, then for every fraction of the form $k/b_j \neq 0$ there exists at least one interval $J_l^{(j')}$ which is not centered in it.

From (2.13) we can see that

$$(2.14) \quad \int_{\mathcal{S}} \ll N^{\frac{m}{3}}, \quad \left(\int_{\frac{Q}{N}}^{\frac{1}{bN^{1/3}}} + \int_{-\frac{1}{bN^{1/3}}}^{-\frac{Q}{N}} \right) \ll \frac{1}{b_1 \dots b_m} \int_{\frac{Q}{N}}^{\frac{1}{bN^{1/3}}} \frac{dy}{y^m} \ll \frac{1}{b_1 \dots b_m} \left(\frac{N}{Q} \right)^{m-1},$$

$$\sum_{j=1}^m \sum_{k=1}^{b_j-1} \left| \int_{I_k^{(j)}} + \int_{J_k^{(j)} \setminus I_k^{(j)}} \right| \ll \sum_{j=1}^m \sum_{k=1}^{b_j-1} \left(N^{m-1+\frac{1}{3}} \frac{1}{b_j N^{1/2}} + N^{\frac{m-1}{2}+\frac{1}{3}} \frac{1}{b_j N^{1/3}} \right) \ll N^{(1+\delta)m-\frac{7}{6}}.$$

Combining (2.11), (2.12), (2.13), (2.14) and recalling the assumption $\delta < \frac{1}{12m}$, we conclude that

$$(2.15) \quad \int_{\mathbf{R}/\mathbf{Z}} \prod_{i=1}^m u_i(y) e(-Ny) dy = J_{b_1, \dots, b_m}(N) + \frac{1}{b_1 \dots b_m} O\left(\left(\frac{N}{Q}\right)^{m-1}\right)$$

From (2.8), (2.9), (2.15) and Theorem A.2 we get

$$(2.16) \quad \int_{\mathcal{M}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha = \frac{1}{(m-1)! b_1 \dots b_m} \mathfrak{S}_{c_1, \dots, c_m}(N) + \frac{1}{b_1 \dots b_m} O\left(\frac{N^{m-1}}{Q^{m-2-\varepsilon}}\right).$$

□

We end the discussion in this section by proving a useful lemma on the function $\mathfrak{S}_{c_1, \dots, c_m}$:

Lemma 2.2. *If $(c_1, \dots, c_m) = 1$, then it is true that $\mathfrak{S}_{c_1, \dots, c_m}(N) \neq 0$ iff $c_1 + \dots + c_m + N \equiv 0 \pmod{2}$ and*

$$(N, c_2, \dots, c_m) = \dots = (c_1, \dots, c_{i-1}, N, c_{i+1}, \dots, c_m) = \dots = (c_1, \dots, c_{m-1}, N) = 1.$$

Moreover, $\mathfrak{S}_{c_1, \dots, c_m}(N) \gg 1$ for every N such that $\mathfrak{S}_{c_1, \dots, c_m}(N) \neq 0$.

Proof. The first part of the lemma follows almost directly from the definition of $\mathfrak{S}_{c_1, \dots, c_m}$.

The second part follows from the fact that there exist at most finitely many primes which divide $\prod_{i=1}^m c_i$. Let z be a real number which is greater than all of them. From the first part of this lemma one can see that for N such that $\mathfrak{S}_{c_1, \dots, c_m}(N) \neq 0$ we have

$$\mathfrak{S}_{c_1, \dots, c_m}(N) \geq 2 \prod_{\substack{p|N \\ 2 < p \leq z}} \left(1 - \frac{1}{p-1}\right) \prod_{\substack{p \nmid N \\ 2 < p \leq z}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p > z} \left(1 - \frac{1}{(p-1)^{m-1}}\right) \gg 1 \quad \square$$

3. MINOR ARCS

In this section we will estimate the integral

$$(3.1) \quad \int_{\mathfrak{m}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha.$$

Usually, some variations of the Vinogradov Lemma (lemma 3.4 in our case) are used to establish results of this type. Recall the following result which is going to be helpful further:

Lemma 3.1 (R. C. Vaughan [3]). *Let $X, Y, \alpha \in \mathbf{R}$ where $X, Y \geq 1$. Let us assume that $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$ for some $a, q \in \mathbf{N}$ such that $(a, q) = 1$. Then*

$$\sum_{n \leq X} \min \left\{ \frac{XY}{n}, \frac{1}{\|n\alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} \ll \left(\frac{XY}{q} + X + q \right) \log(2Xq).$$

We will also need the following version of the Vaughan's identity:

Lemma 3.2. *For every real $x > 0$, $U, V \geq 2$ we have the identity*

$$S(x, \alpha) = S_{I,1} - S_{I,2} - S_{II} + S_0,$$

where

$$\begin{aligned} S_{I,1} &= \sum_{d \leq U} \mu(d) \sum_{n \leq \frac{x}{d}} \log ne(nd\alpha), \\ S_{I,2} &= \sum_{d \leq V} \Lambda(d) \sum_{\delta \leq U} \mu(\delta) \sum_{n \leq \frac{x}{d\delta}} e(nd\delta\alpha), \\ S_{II} &= \sum_{d > U} \left(\sum_{\substack{\delta \leq U \\ \delta|d}} \mu(\delta) \right) \sum_{\substack{n > V \\ nd \leq x}} \Lambda(n) e(nd\alpha), \\ S_0 &= \sum_{n \leq V} \Lambda(n) e(n\alpha). \end{aligned}$$

Let us use Lemma 3.2 by putting $x = \frac{N}{b_i}$ and $b_i\alpha$ instead of α . The method used here is well described in [3], however we will present it for the sake of completeness.

The inner sum in $S_{I,1}$ is equal to

$$\sum_{n \leq \frac{N}{db_i}} e(ndb_i\alpha) \int_1^n \frac{dy}{y} = \int_1^{\frac{N}{db_i}} \sum_{n \leq \frac{N}{db_i}} e(ndb_i\alpha) \mathbf{1}_{y < n} \frac{dy}{y} = \int_1^{\frac{N}{db_i}} \sum_{y < n \leq \frac{N}{db_i}} e(ndb_i\alpha) \frac{dy}{y}$$

which gives us

$$(3.2) \quad S_{I,1} = \sum_{d \leq U} \mu(d) \int_1^{\frac{N}{db_i}} e(\lceil y \rceil d\alpha) \sum_{n \leq \frac{N}{db_i} - y} e(ndb_i\alpha) \frac{dy}{y} \ll \log N \sum_{d \leq U} \min \left\{ \frac{N}{db_i}, \frac{1}{\|db_i\alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} = \log N \sum_{\substack{d \leq U \\ b_i | d}} \min \left\{ \frac{N}{d}, \frac{1}{\|d\alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\}.$$

Moreover, one can see that

$$(3.3) \quad S_{I,2} = \sum_{\delta_1 \leq U, \delta_2 \leq V} \sum_{d \leq \frac{N}{\delta_1 \delta_2 b_i}} \mu(\delta_1) \Lambda(\delta_2) e(\delta_1 \delta_2 db_i \alpha) = \sum_{\delta_1 \leq U, \delta_2 \leq V} \sum_{d \leq \frac{N}{\delta_1 \delta_2 b_i}} \mu(\delta_1) \Lambda(\delta_2) e(ndb_i \alpha) = \sum_{\substack{\delta_1 \leq U, \delta_2 \leq V, d, n \leq UV \\ \delta_1 \delta_2 = n, dn \leq \frac{N}{b_i}}} \mu(\delta_1) \Lambda(\delta_2) e(ndb_i \alpha) = \sum_{n \leq UV} \left(\sum_{\substack{\delta_1 \leq U, \delta_2 \leq V \\ \delta_1 \delta_2 = n}} \mu(\delta_1) \Lambda(\delta_2) \right) \sum_{d \leq \frac{N}{nb_i}} e(ndb_i \alpha) \ll \log UV \sum_{n \leq UV} \min \left\{ \frac{N}{nb_i}, \frac{1}{\|nb_i \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} = \log UV \sum_{\substack{d \leq UV \\ b_i | d}} \min \left\{ \frac{N}{d}, \frac{1}{\|d\alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\}.$$

To estimate the value of S_{II} in a similar manner let us define the set

$$\mathcal{Y} := \{U, 2U, 4U, \dots, 2^k U : 2^k UV < N/b_i \leq 2^{k+1} UV\}.$$

Thus

$$S_{II} = \sum_{Z \in \mathcal{Y}} S(Z), \quad \text{where} \quad S(Z) = \sum_{Z < d \leq 2Z} \left(\sum_{\substack{\delta \leq U \\ \delta | d}} \mu(\delta) \right) \sum_{V < n \leq \frac{N}{db_i}} \Lambda(n) e(ndb_i \alpha).$$

By the Cauchy-Schwarz inequality

$$|S(Z)|^2 \leq \sum_{Z < d \leq 2Z} \tau(d)^2 \sum_{Z < d \leq 2Z} \left| \sum_{V < n \leq \frac{N}{db_i}} \Lambda(n) e(ndb_i \alpha) \right|^2.$$

From $\sum_{n \leq x} \tau(n)^2 \ll n \log^3 n$ for $n \geq 2$ and the following identity

$$\left| \sum_{V < n \leq \frac{N}{db_i}} \Lambda(n) e(ndb_i \alpha) \right|^2 = \sum_{V < n_1, n_2 \leq \frac{N}{db_i}} \Lambda(n_1) \Lambda(n_2) e((n_1 - n_2)db_i \alpha),$$

one gets

$$(3.4) \quad |S(Z)|^2 \ll Z \log^3 N \sum_{V < n_1, n_2 \leq \frac{N}{db_i}} \Lambda(n_1) \Lambda(n_2) \sum_{Z < d \leq 2Z} e((n_1 - n_2)db_i \alpha) \ll$$

$$Z \log^5 N \sum_{n_1, n_2 \leq \frac{N}{db_i}} \min \left\{ Z, \frac{1}{\|(n_1 - n_2)b_i \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} =$$

$$Z \log^5 N \sum_{\substack{n \leq \frac{N}{Z} \\ b_i | n}} \sum_{\substack{-n \leq d \leq \frac{N}{Z} - n \\ b_i | d}} \min \left\{ Z, \frac{1}{\|d \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} \ll \frac{N}{b_i} \log^5 N \left(Z + \sum_{\substack{d \leq \frac{N}{Z} \\ b_i | d}} \min \left\{ \frac{N}{d}, \frac{1}{\|d \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} \right),$$

Let us combine (3.2), (3.3), (3.4) and put $U, V := \left(\frac{N}{b_i}\right)^{\frac{2}{5}}$. Let us say that $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$ for some $a, q \in \mathbf{N}$ such that $(a, q) = 1$, $q \leq N$. Using Lemma 3.1 one gets²

$$(3.5) \quad \sum_{d \leq Ub_i} \min \left\{ \frac{N}{d}, \frac{1}{\|d \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\}, \quad \sum_{n \leq UVb_i} \min \left\{ \frac{N}{n}, \frac{1}{\|n \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} \ll \left(\frac{N}{q} + N^{\frac{4}{5}} b_i^{\frac{1}{5}} + q \right) \log N,$$

$$Z + \sum_{d \leq \frac{N}{Z}} \min \left\{ \frac{N}{d}, \frac{1}{\|d \alpha\|_{\mathbf{R}/\mathbf{Z}}} \right\} \ll \left(\frac{N}{q} + \frac{N}{Z} + q + Z \right) \log N.$$

Since

$$|S(Z)|^2 \ll \frac{N}{b_i} \log^6 N \left(\frac{N}{q} + \frac{N}{Z} + q + Z \right),$$

one gets

$$S_{II} \ll \frac{1}{\sqrt{b_i}} \sum_{Z \in \mathcal{Y}} \log^3 N \left(\frac{N}{\sqrt{q}} + \frac{N}{\sqrt{Z}} + \sqrt{Nq} + \sqrt{NZ} \right) \ll \frac{1}{\sqrt{b_i}} \log^4 N \left(\frac{N}{\sqrt{q}} + N^{\frac{4}{5}} + \sqrt{Nq} \right).$$

The best result we are able to obtain is the following estimate.

²Note that the estimations are rather weak, especially if we want to use them to (3.2), (3.3) and (3.4) sums. The restrictions on $b_i | d$ under every summand were simply cancelled. The reason behind such a manouver is the lack of any visible possibility to use them. On the other hand, one has some chance to improve these results, some other ideas are required though.

Lemma 3.3. *Let $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$ for some $a, q \in \mathbf{N}$ such that $(a, q) = 1$, $q \leq N$. Then*

$$S_i(N, \alpha) \ll \log^4 N \left(\frac{N}{\sqrt{q}} + N^{\frac{4}{5}} + \sqrt{Nq} \right).$$

Proof. We use the bounds on $S_{I,1}$, $S_{I,2}$ i S_{II} and the obvious fact that $S_0 \ll N^{\frac{2}{5}}$. \square

Using the fact that for any $\alpha \in \mathbf{R}$ there exists a positive integer $q \leq \frac{N}{Q}$ such that $|\alpha - \frac{a}{q}| \leq \frac{Q}{qN} \leq \frac{1}{q^2}$ for some $a \in \mathbf{N}$ which fulfils the condition $(a, q) = 1$ we get

Lemma 3.4. *Let $B > 0$ and a positive integer $b_i \leq N^{\frac{1}{36}}$. Hence for every $\alpha \in \mathfrak{m}$ we have*

$$S_i(N, \alpha) \ll \frac{N}{\log^{\frac{B}{2}-4} N}.$$

We shall prove the following

Theorem 3.5. *Under assumptions of the Theorem (2.1) and the extra assumption $\eta_1 = \eta_2 = \eta_3 = 1$ we have*

$$\int_{\mathfrak{m}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha \ll \frac{1}{b_1 \dots b_m} \frac{N^{m-1}}{\log^{\frac{B}{2}-6} N}.$$

Proof. We have

$$(3.6) \quad \int_{\mathfrak{m}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) d\alpha \leq \prod_{i=3}^m \max_{\alpha \in \mathfrak{m}} |S_i(N, \alpha)| \int_{\mathbf{R}/\mathbf{Z}} |S_1(N, \alpha) S_2(N, \alpha)| d\alpha \leq$$

$$\frac{N^{m-3}}{b_4 \dots b_m} \times \max_{\alpha \in \mathfrak{m}} |S_3(N, \alpha)| \left(\int_{\mathbf{R}/\mathbf{Z}} |S_1(N, \alpha)|^2 d\alpha \right)^{1/2} \left(\int_{\mathbf{R}/\mathbf{Z}} |S_2(N, \alpha)|^2 d\alpha \right)^{1/2} \ll$$

$$\frac{N^{m-3}}{b_4 \dots b_m} \frac{N}{\log^{\frac{B}{2}-4} N} \frac{N}{\sqrt{b_1 b_2}} \log(N/b_1) \log(N/b_2) \ll \frac{N^{m-1}}{b_1 \dots b_m} \frac{1}{\log^{\frac{B}{2}-6} N},$$

which follows from the Cauchy-Schwarz inequality, the Plancherel identity, the constancy of b_1, b_2, b_3 and the basic fact that $\sum_{n \leq x} \Lambda(n)^2 \ll x \log x$ for $x \geq 1$. \square

Right now we can combine the results from last two sections to establish the following

Theorem 3.6. *Let us consider the constants $c_1, \dots, c_m \in \mathbf{N}$. Let $b_1, \dots, b_m \leq N^\delta$ for some $\delta \in (0, \frac{1}{12m})$ and let $b_i = c_i \eta_i$ for every $i = 1, \dots, m$ where the η_i are some positive integers all of whose prime divisors are greater than Q . Let us further assume that $(b_1, \dots, b_m) = 1$ and $\eta_1, \eta_2, \eta_3 = 1$. Hence for some $A > 0$ we have*

$$\sum_{\substack{n_1, \dots, n_m \leq N \\ b_1 n_1 + \dots + b_m n_m = N}} \Lambda(n_1) \dots \Lambda(n_m) = \frac{1}{(m-1)!} \frac{N^{m-1}}{b_1 \dots b_m} \mathfrak{S}_{c_1, \dots, c_m}(N) + \frac{1}{b_1 \dots b_m} O\left(\frac{N^{m-1}}{\log^A N}\right)$$

Proof. Follows easily from 2.1 and 3.3 upon taking $B = 2A + 12$ and $\varepsilon = \frac{1}{2}$. \square

Corollary 3.7. Under assumptions of Theorem 3.6 for sufficiently large N we have $\mathfrak{S}_{c_1, \dots, c_m}(N) = 0$ iff $R_m(N; b_1, \dots, b_m) = 0$.

Proof. Follows from Lemma 2.2 and Theorem 3.6. \square

4. REDUCING THE LOGARITHMIC WEIGHTS

Firstly we are going to show that the contribution of the numbers of the form p^k for some $k \geq 2$ which appear in the support of the von Mangoldt function does not change the asymptotics of $R_m(N; b_1, \dots, b_m)$ given in Theorem 3.6. Let us define

$$(4.1) \quad \theta(n) = \begin{cases} \log n, & \text{if } n \in P \\ 0, & \text{otherwise} \end{cases}.$$

Then for some $A > 0$ one has

$$(4.2) \quad \begin{aligned} & \sum_{\substack{n_1, \dots, n_m \leq N \\ b_1 n_1 + \dots + b_m n_m = N}} \Lambda(n_1) \dots \Lambda(n_m) - \sum_{\substack{n_1, \dots, n_m \leq N \\ b_1 n_1 + \dots + b_m n_m = N}} \theta(n_1) \dots \theta(n_m) \leq \\ & \sum_{i=1}^m \sum_{\substack{n_1, \dots, n_m \leq N \\ b_1 n_1 + \dots + b_m n_m = N \\ \Omega(n_i) \geq 2}} \Lambda(n_1) \dots \Lambda(n_m) \leq \log^m N \sum_{i=1}^m \sum_{\substack{n_1, \dots, n_m \leq N, n_i \leq \sqrt{N} \\ b_1 n_1 + \dots + b_m n_m = N \\ \omega(n_i) = 1, \Omega(n_i) \geq 2}} 1 \leq \\ & \log^m N \sum_{i=1}^m \sum_{k_1 \leq \sqrt{N}, k_2, \dots, k_{m-1} \leq N} 1 = N^{m-\frac{3}{2}} \log^m N \leq \frac{N^{m-\frac{3}{2}+\delta m} \log^m N}{b_1 \dots b_m}. \end{aligned}$$

We are going to study the asymptotics of the function

$$r_m(N; b_1, \dots, b_m) = \sum_{\substack{n_1 \leq \frac{N}{b_1}, \dots, n_m \leq \frac{N}{b_m} \\ b_1 n_1 + \dots + b_m n_m = N}} \mathbf{1}_{b_1 n_1 + \dots + b_m n_m = N} \prod_{l=1}^m \mathbf{1}_{n_l \in P}.$$

We can also define the function

$$\widetilde{R}_m(N; b_1, \dots, b_m) = \sum_{\substack{n_1 \leq \frac{N}{b_1}, \dots, n_m \leq \frac{N}{b_m} \\ b_1 n_1 + \dots + b_m n_m = N}} \mathbf{1}_{b_1 n_1 + \dots + b_m n_m = N} \theta(n_1) \dots \theta(n_m).$$

We showed that $\widetilde{R}_m = \frac{1}{(m-1)!} \frac{N^{m-1}}{b_1 \dots b_m} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1))$. Let us prove the following

Theorem 4.1. Under assumptions of Theorem 3.6 we have

$$r_m(N; b_1, \dots, b_m) \prod_{i=1}^m \log \frac{N}{b_i} = \widetilde{R}_m(N; b_1, \dots, b_m) (1 + o(1)).$$

Proof. Obviously $\widetilde{R}_m = 0$ iff $r_m = 0$ and in such a case the theorem follows trivially. Note that

$$(4.3) \quad \widetilde{R}_m(N; b_1, \dots, b_m) \leq r_m(N; b_1, \dots, b_m) \prod_{i=1}^m \log \frac{N}{b_i}.$$

On the other hand, notice that for every $\epsilon > 0$ one has

$$(4.4) \quad \widetilde{R}_m(N; b_1, \dots, b_m) \geq \sum_{\left(\frac{N}{b_i}\right)^{1-\epsilon} < n_i \leq \frac{N}{b_i}: 1 \leq i \leq m} \mathbf{1}_{b_1 n_1 + \dots + b_m n_m = N} \theta(n_1) \dots \theta(n_m) \geq \\ (1-\epsilon)^m \prod_{i=1}^m \log \frac{N}{b_i} \times \sum_{\left(\frac{N}{b_i}\right)^{1-\epsilon} < n_i \leq \frac{N}{b_i}: 1 \leq i \leq m} \mathbf{1}_{b_1 n_1 + \dots + b_m n_m = N} \prod_{l=1}^m \mathbf{1}_{n_l \in P}.$$

The last sum in (4.4) differs from r_m only by

$$\ll \sum_{j=1}^m \sum_{\substack{n_i \leq \frac{N}{b_i}: 1 \leq i \leq m \\ n_j \leq \left(\frac{N}{b_j}\right)^{1-\epsilon}}} \mathbf{1}_{b_1 n_1 + \dots + b_m n_m = N} \ll \sum_{j=1}^m \frac{N^{m-1-\epsilon} b_j^\epsilon}{b_1 \dots b_m} \ll \frac{N^{m-1-(1-\delta)\epsilon}}{b_1 \dots b_m},$$

because the restriction under the indicator annihilates one of the variables in a natural way and we are always able to choose one of n_i fulfilling $b_i = c_i$ in such a role. In the next step we can multiply the error term by $1/b_i$ without any repercussions. Note that this estimation is in fact trivial because the primality of n_1, \dots, n_m was not used³. Now from (4.4) and the corollary 3.7 we get

$$(4.5) \quad 1 \geq (1-\epsilon)^m \frac{r_m(N; b_1, \dots, b_m)}{\widetilde{R}_m(N; b_1, \dots, b_m)} \prod_{i=1}^m \log \frac{N}{b_i} + O\left(\frac{1}{N^{(1-\delta)\epsilon}}\right).$$

for every N which satisfies $\widetilde{R}_m(N; b_1, \dots, b_m) \neq 0$. We have (4.5) working for every $\epsilon > 0$ which finishes the proof. \square

Corollary 4.2. Under assumptions of Theorem 3.6 we have

$$r_m(N; b_1, \dots, b_m) = \frac{1}{(m-1)!} \frac{1}{b_1 \dots b_m} \frac{N^{m-1}}{\prod_{i=1}^m \log \frac{N}{b_i}} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)).$$

5. CUTTING OFF

Recall that $(c_1, \dots, c_m) = 1$ and let N be large enough to have $Q := \log^B N > \max\{c_1, \dots, c_m\}$. From this point we assume that the $n_j^{(i)}$ are always prime so this fact will not be emphasized under summands. In this section we will deal with the sum

$$(5.1) \quad \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i=1}^{r_m} n_m^{(i_m)} = N}.$$

We are going to show that even after attaching some stronger conditions to the sum (5.1), the asymptotic will not change. Let $r = \max\{r_1, \dots, r_m\}$. A set of numbers which appears in the summation as $n_j^{(i)}$ for $1 \leq i \leq m$ and $2 \leq j \leq r_i$ will be cut from $[1, \sqrt{N}] \cap \mathbf{N}$ to $[Q, N^{\frac{1}{24mr}}] \cap \mathbf{N}$. These restrictions are sufficient to calculate the asymptotic of the sum by using Corollary 4.2. Therefore (5.1) equals

³In the same time it is the next situation in which the constancy of at least two of b_i 's was used, although it is possible to write a little bit longer proof based only on properties of $J_{b_1, \dots, b_m}(N)$.

$$(5.2) \quad \sum^\dagger + O\left(\sum_{\ell=1}^m \sum_{k=2}^{r_\ell} \left(\sum_1^{(\ell,k)} + \sum_2^{(\ell,k)}\right) + \sum_3\right),$$

where

$$\begin{aligned} \sum^\dagger &= \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ (\eta_1, \dots, \eta_m) = 1}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N}, \\ \sum_1^{(\ell,k)} &= \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N} : 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ n_\ell^{(k)} > N^{\frac{1}{24mr}}}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N}, \\ \sum_2^{(\ell,k)} &= \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq N^{\frac{1}{24mr}} : 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ n_\ell^{(k)} \leq Q}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N}, \\ \sum_3 &= \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ (\eta_1, \dots, \eta_m) > 1}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N}. \end{aligned}$$

Our assumption $\eta_1 = \eta_2 = \eta_3 = 1$ gives

$$\sum_3 = 0.$$

5.1. Estimating \sum^\dagger . For the sake of simplicity we will use the notation $n_j = \prod_{i_j=1}^{r_j} n_j^{(i_j)}$ and $\eta_j = \prod_{i_j=2}^{r_j} n_j^{(i_j)}$ for $j = 1, \dots, m$. We can rewrite the sum in the following form

$$\sum^\dagger = \sum_{\substack{Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : n_1^{(1)} \leq \frac{N}{c_1 \eta_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m \eta_m} \\ 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ (\eta_1, \dots, \eta_m) = 1}} \sum_{(\eta_1, \dots, \eta_m) = 1} \mathbf{1}_{c_1 \eta_1 n_1^{(1)} + \dots + c_m \eta_m n_m^{(1)} = N}.$$

We have $\eta_j \leq N^{\frac{1}{24m}}$ and $c_j \leq N^{\frac{1}{24m}}$ for $j = 1, \dots, m$ and N sufficiently large which gives $c_j \eta_j \leq N^{\frac{1}{12m}}$. The sum in parentheses equals to $r_m(N; c_1 \eta_1, \dots, c_m \eta_m)$, and then the condition $(\eta_1, \dots, \eta_m) = 1$ is sufficient to enable us able to use Corollary 4.2 here. Hence (from $\eta_1, \eta_2, \eta_3 = 1$) we can transform the right hand side of the equality above to the form

$$(5.3) \quad \frac{1}{(m-1)!} \frac{N^{m-1}}{c_1 \dots c_m} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)) \times \\ \sum_{Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : 1 \leq j \leq m, \ 2 \leq i \leq r_j} \frac{1}{\eta_1 \dots \eta_m} \frac{1}{\log \frac{N}{c_1 \eta_1} \dots \log \frac{N}{c_m \eta_m}} \mathbf{1}_{(\eta_1, \dots, \eta_m) = 1}.$$

Given that $\mathbf{1}_{(\eta_1, \dots, \eta_m) = 1} \leq 1$ we can bound it from above by

$$\prod_{j=1}^m \sum_{Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : 2 \leq i \leq r_j} \frac{1}{\eta_j} \frac{1}{\log \frac{N}{c_j \eta_j}} =$$

$$\prod_{j=1}^m \sum_{Q < n_j^{(r_j)} \leq N^{\frac{1}{24mr}}} \left(\cdots \left(\sum_{Q < n_j^{(3)} \leq N^{\frac{1}{24mr}}} \left(\sum_{Q < n_j^{(2)} \leq N^{\frac{1}{24mr}}} \frac{1}{n_j^{(2)}} \frac{1}{\log \frac{N}{c_j \eta_j}} \right) \frac{1}{n_j^{(3)}} \right) \cdots \right) \frac{1}{n_j^{(r_j)}}.$$

From $\frac{N}{c_j \eta_j} < N^{1-\frac{1}{12m}}$ for sufficiently large N and then from Lemma B.2 used $r_j - 1$ times we get that the expression above is equal to

$$(5.4) \quad (1 + o(1)) \prod_{j=1}^m \frac{(\log \log N)^{r_j-1}}{\log \frac{N}{c_j}} = (1 + o(1)) \frac{(\log \log N)^{r_1 + \cdots + r_m - m}}{\log^m N}.$$

Estimating from below the sum from the equation (5.3) bases on the simple inequality $\mathbf{1}_{(\eta_1, \dots, \eta_m)=1} \geq \mathbf{1}_{(\eta_s, \eta_{s'})=1 : s, s'=1, \dots, m, s \neq s'}$ and $\log \frac{N}{c_j \eta_j} \leq \log N$. Right now we are able to bound the sum as follows:

$$(5.5) \quad \frac{1}{\log^m N} \sum_{Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : 1 \leq j \leq m, 2 \leq i \leq r_j} \frac{1}{\eta_1 \cdots \eta_m} \prod_{1 \leq s' < s \leq m} \mathbf{1}_{(\eta_{s'}, \eta_s)=1} =$$

$$\frac{1}{\log^m N} \sum_{\substack{Q < n_m^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_m}} \left(\cdots \left(\sum_{\substack{Q < n_1^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_1}} \frac{1}{\eta_1 \cdots \eta_m} \prod_{1 \leq s' < s \leq m} \mathbf{1}_{(\eta_{s'}, \eta_s)=1} \right) \cdots \right) =$$

$$\frac{1}{\log^m N} \sum_{\substack{Q < n_m^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_m}} \left(\cdots \left(\sum_{\substack{Q < n_1^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_1}} \frac{1}{\eta_1} \prod_{1 \leq s \leq m} \mathbf{1}_{(\eta_1, \eta_s)=1} \right) \cdots \right) \times$$

$$\frac{1}{\eta_m} \prod_{m < s \leq m} \mathbf{1}_{(\eta_m, \eta_s)=1}.$$

We interpret the empty product simply as being equal to 1 (it is presented only to show the pattern in interchanging the terms). The restriction represented by the product of indicators of the form $\mathbf{1}_{(\eta_{s'}, \eta_s)=1}$ forces us to omit at most $rm = O(1)$ terms in each summation. From Bertrand's postulate we know that there exists at least one prime in the interval $(Q, 2Q]$. Iterating the argument rm times we can say that there exists at least rm primes in $(Q, 2^{rm}Q]$. Our summations are defined over the primes (thanks to the indicator) from $(Q, N^{\frac{1}{24m}}]$, so we can estimate the expression (5.5) from below by

$$(5.6) \quad \frac{1}{\log^m N} \sum_{\substack{2^{rm}Q < n_m^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_m}} \left(\cdots \left(\sum_{\substack{2^{rm}Q < n_1^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_1}} \frac{1}{\eta_1} \right) \cdots \right) \frac{1}{\eta_m} =$$

$$\frac{1}{\log^m N} \sum_{\substack{2^{rm} Q < n_j^{(i)} \leq N^{\frac{1}{24mr}} : \\ 2 \leq i \leq r_j, 1 \leq j \leq m}} \frac{1}{\eta_1 \dots \eta_m} = \frac{1}{\log^m N} \left(\sum_{2^{rm} Q < p \leq N^{\frac{1}{24mr}}} \frac{1}{p} \right)^{r_1 + \dots + r_m - m} =$$

$$(1 + o(1)) \frac{(\log \log N)^{r_1 + \dots + r_m - m}}{\log^m N}$$

by the Mertens' theorem. Combining (5.3), (5.4) and (5.6) we finally conclude

$$(5.7) \quad \sum^\dagger = \frac{1}{(m-1)!} \frac{N^{m-1}}{c_1 \dots c_m} \frac{(\log \log N)^{r_1 + \dots + r_m - m}}{\log^m N} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)).$$

5.2. Upper bounds on $\sum_1^{(\ell, k)}$ and $\sum_2^{(\ell, k)}$. We will deal with these two sums in exactly the same way, so only the calculations for the first one will be presented in details. We will assume that $\ell \geq 4$.

The first sum can be rewritten in the following manner

$$\sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ n_\ell^{(k)} > N^{\frac{1}{24mr}}}} \mathbf{1}_{c_1 \eta_1 n_1^{(1)} + \dots + c_m \eta_m n_m^{(1)} = N} =$$

$$\sum_{\substack{n_4^{(1)} \leq \frac{N}{c_4}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, \ 2 \leq i \leq r_j \\ c_j \eta_j \leq \frac{N}{2} \\ n_\ell^{(k)} > N^{\frac{1}{24mr}}}} \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, n_2^{(1)} \leq \frac{N}{c_2}, n_3^{(1)} \leq \frac{N}{c_m}}} \mathbf{1}_{c_1 \eta_1 n_1^{(1)} + \dots + c_3 \eta_3 n_3^{(1)} = N - c_4 \eta_4 n_4^{(1)} - \dots - c_m \eta_m n_m^{(1)}}.$$

By $\eta_1 = \eta_2 = \eta_3 = 1$ and Lemma B.1, the term inside the bracket is

$$R_m(N - c_4 \eta_4 n_4^{(1)} - \dots - c_m \eta_m n_m^{(1)}; c_1, c_2, c_3) \ll \frac{N^2}{\log^3 N}$$

Thus we have

$$\sum_1^{(\ell, k)} \ll \frac{N^2}{\log^3 N} \sum_{\substack{n_4^{(1)} \leq \frac{N}{c_4}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 4 \leq j \leq m, \ 2 \leq i \leq r_j \\ n_j \leq N \\ n_\ell^{(k)} > N^{\frac{1}{24mr}}}} 1 \ll$$

$$\frac{N^2}{\log^3 N} \sum_{\substack{n_j \leq N: \\ 4 \leq j \leq m, j \neq \ell}} \prod_{\substack{l=4 \\ l \neq \ell}}^m \mathbf{1}_{\Omega(n_l)=r_l} \times \sum_{\substack{n_\ell^{(1)}, \dots, n_\ell^{(r_\ell)} \leq N \\ n_\ell \leq N \\ N^{\frac{1}{24mr}} < n_\ell^{(k)} \leq \sqrt{N}}} 1.$$

From Theorem B.3 and Mertens' theorem we have

$$\begin{aligned}
\sum_{\substack{n_\ell^{(1)}, \dots, n_\ell^{(r_\ell)} \leq N \\ n_\ell \leq N \\ N^{\frac{1}{24mr}} < n_\ell^{(k)} \leq \sqrt{N}}} 1 &\ll \sum_{N^{\frac{1}{24mr}} < n_\ell^{(k)} \leq \sqrt{N}} \sum_{h \leq \frac{N}{n_\ell^{(k)}}} \mathbf{1}_{\Omega(h)=r_\ell-1} \ll \\
N(\log \log N)^{r_\ell-2} \sum_{N^{\frac{1}{24mr}} < n_\ell^{(k)} \leq \sqrt{N}} \frac{1}{n_\ell^{(k)} \log \frac{N}{n_\ell^{(k)}}} &\ll \frac{N(\log \log N)^{r_\ell-2}}{\log N}
\end{aligned}$$

and for $4 \leq j \neq \ell$

$$\sum_{n_j \leq N} \mathbf{1}_{\Omega(n_j)=r_j} \ll \frac{N(\log \log N)^{r_j-1}}{\log N},$$

which gives

$$\sum_1^{(\ell, k)} \ll \frac{N^{m-1}(\log \log N)^{r_1+\dots+r_m-m-1}}{\log^m N}.$$

Analogously we can obtain the following upper bound for the second sum

$$\sum_2^{(\ell, k)} \ll \frac{N^{m-1}(\log \log N)^{r_1+\dots+r_m-m-1}(\log \log \log N)}{\log^m N}.$$

The $\log \log \log N$ term appears because $n_\ell^{(k)}$ appears in the summation as an index supported on $[1, Q] \cap \mathbf{Z}$ instead of $(N^{1/24mr}, N^{1/2}] \cap \mathbf{Z}$ like in the first case; by Mertens' theorem we easily get

$$\sum_{n_\ell^{(k)} \leq Q} \frac{1}{n_\ell^{(k)} \log \frac{N}{n_\ell^{(k)}}} \ll \frac{\log \log Q}{\log N} \ll \frac{\log \log \log N}{\log N}.$$

5.3. Proof of Theorem 1.2. Right now we can say that under assumptions of Theorem 1.2 we have

$$\begin{aligned}
(5.8) \quad &\sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, \ 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i=1}^{r_m} n_m^{(i_m)} = N} = \\
&\frac{1}{(m-1)!} \frac{N^{m-1}}{c_1 \dots c_m} \frac{(\log \log N)^{r_1+\dots+r_m-m}}{\log^m N} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)).
\end{aligned}$$

and we are ready to finally finish the whole proof.

We should get over the $n_j^{(i)} \leq \sqrt{N}$ restriction in the sum above. Note that for every η_j such that $j = 1, \dots, m$ there can be at most one term greater than \sqrt{N} , hence

$$(5.9) \quad \sum_{n_j^{(i)} \leq \frac{N}{c_j}: 1 \leq j \leq m, \ 1 \leq i \leq r_j} \mathbf{1}_{c_1 \prod_{i=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i=1}^{r_m} n_m^{(i_m)} = N} =$$

$$r_1 \dots r_m \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, \ 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N} +$$

$$O \left(\sum_{K=1}^m \sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_K^{(1)} \leq \sqrt{N}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, \ 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N} \right).$$

Let us further assume that $\eta_1 = \eta_2 = \eta_3 = 1$. The main term in the expression above equals to

$$(5.10) \quad \frac{r_1 \dots r_m}{(m-1)! c_1 \dots c_m} \frac{N^{m-1} (\log \log N)^{r_1 + \dots + r_m - m}}{\log^m N} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)).$$

To study the error term in (5.9) let us fix some $K \in \{1, \dots, m\}$ from the first sum. Then

$$\sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_K^{(1)} \leq \sqrt{N}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ n_j^{(i)} \leq \sqrt{N}: 1 \leq j \leq m, \ 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N} =$$

$$\sum_{\substack{n_1^{(1)} \leq \frac{N}{c_1}, \dots, n_K^{(1)} \leq \sqrt{N}, \dots, n_m^{(1)} \leq \frac{N}{c_m} \\ Q < n_j^{(i)} \leq N^{\frac{1}{24mr}}: 1 \leq j \leq m, \ 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N} +$$

$$O \left(\sum_{\ell=1}^m \sum_{k=2}^{r_\ell} \left(\sum_1^{(\ell, k)} + \sum_2^{(\ell, k)} \right) \right).$$

The 'big O' term has order as big as the error term from (5.10). The main term (which is essentially only the error term in (5.9)) can be bounded by

$$\ll \sum_{\substack{n_1^{(1)} \leq N, \dots, n_K^{(1)} \leq \sqrt{N}, \dots, n_m^{(1)} \leq N \\ Q < n_j^{(i)} \leq N^{\frac{1}{24mr}}: 1 \leq j \leq m, \ 2 \leq i \leq r_j}} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N} \leq N^{m - \frac{3}{2} + \frac{1}{24}}.$$

Combining the results from this subsection we can state that

$$(5.11) \quad \sum_{n_j^{(i)} \leq \frac{N}{c_j}: 1 \leq j \leq m, \ 1 \leq i \leq r_j} \mathbf{1}_{c_1 \prod_{i_1=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i_m=1}^{r_m} n_m^{(i_m)} = N} =$$

$$\frac{r_1 \dots r_m}{(m-1)! c_1 \dots c_m} \frac{N^{m-1} (\log \log N)^{r_1 + \dots + r_m - m}}{\log^m N} (\mathfrak{S}_{c_1, \dots, c_m}(N) + o(1)).$$

It is also worth mentioning that from Lemma B.1 and Theorem B.3 we can repeat the trick from the previous subsection to obtain the following upper bound for some $1 \leq \ell \leq m$ and $1 \leq k_1, k_2 \leq r_\ell$ for which $k_1 \neq k_2$:

$$(5.12) \quad \sum_{\substack{n_j^{(i)} \leq \frac{N}{c_j} : 1 \leq j \leq m, 1 \leq i \leq r_j \\ n_\ell^{(k_1)} = n_\ell^{(k_2)}}} \mathbf{1}_{c_1 \prod_{i=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i=1}^{r_m} n_m^{(i_m)} = N} \ll \frac{N^2}{\log^3 N} \sum_{\substack{n_j^{(i)} \leq \frac{N}{c_j} : 1 \leq j \leq m, 1 \leq i \leq r_j \\ n_j \leq N \\ n_\ell^{(k_1)} = n_\ell^{(k_2)}}} 1 \ll \frac{N^{m-1} (\log \log N)^{r_1 + \dots + r_m - m - 1}}{\log^m N}.$$

Therefore, we have the following identity

$$(5.13) \quad \sum_{n_j^{(i)} \leq \frac{N}{c_j} : 1 \leq j \leq m, 1 \leq i \leq r_j} \mathbf{1}_{c_1 \prod_{i=1}^{r_1} n_1^{(i_1)} + \dots + c_m \prod_{i=1}^{r_m} n_m^{(i_m)} = N} = r_1! \dots r_m! \sum_{\substack{n_1, \dots, n_m \leq N \\ c_1 n_1 + \dots + c_m n_m = N}} \prod_{i=1}^m \mathbf{1}_{\Omega(n_i) = r_i}$$

which finally finishes the proof of the Theorem 1.2.

APPENDIX A.

Let us define $J_{b_1, \dots, b_m}(N)$ as the number of tuples $(n_1, \dots, n_m) \in \mathbf{N}^m$ which obey $b_1 n_1 + \dots + b_m n_m = N$ for some $b_1, \dots, b_m \in \mathbf{N}$. We also define a lattice to be a submodule L of \mathbf{Z}^m over \mathbf{Z} . Every lattice can be represented as a set of the form

$$L = \{a_1 \mathbf{v}_1 + \dots + a_K \mathbf{v}_K : a_1, \dots, a_K \in \mathbf{Z}\}$$

for some $\mathbf{v}_1, \dots, \mathbf{v}_K \in \mathbf{Z}^K$ where addition and multiplication are defined in an obvious manner. Then $\{\mathbf{v}_j\}_{j=1}^K$ will be called basis of the lattice and the set

$$\{t_1 \mathbf{v}_1 + \dots + t_K \mathbf{v}_K : t_1, \dots, t_K \in [0, 1)\}$$

minimal parallelogram of the lattice. Neither basis nor minimal parallelogram are unique although the K -dimensional measure of this parallelogram is and we will call it a determinant of the lattice and denote it by $d(L)$.

Recall the following

Theorem A.1. *Every lattice L admits a basis $\{\mathbf{v}_j\}_{j=1}^K$ such that*

$$\prod_{j=1}^K \|\mathbf{v}_j\| \ll_K d(L).$$

The proof can be found in [8]. The constant can be made explicit, for example for $K \geq 5$ one has

$$\left(\frac{2}{\pi}\right)^{-K} \Gamma\left(\frac{K+1}{2}\right)^{-1} \left(\frac{4}{5}\right)^{\frac{1}{2}(K-3)(K-4)},$$

but such a strong statement is not necessary for our purposes.

A.1. Proof of geometric lemma.

Lemma A.2. *Let $\delta \in \mathbf{R}_+$. Take the positive integers $b_1, \dots, b_m \leq N^\delta$ such that $(b_1, \dots, b_m) = 1$. Then*

$$J_{b_1, \dots, b_m}(N) = \frac{N^{m-1}}{(m-1)!} \frac{1}{b_1 \dots b_m} + O\left(N^{2\delta(m-1)+m-2}\right).$$

Proof. Let

$$\begin{aligned} \Lambda &:= \mathbf{Z}^m, \\ \tilde{\Lambda} &:= \{(n_1, \dots, n_m) \in \mathbf{Z}^m : n_1 + \dots + n_m = 0\}; \end{aligned}$$

this sets equipped with obvious actions are lattices.

We can transform the condition $b_1 n_1 + \dots + b_m n_m = 0$ into $n_1 + \dots + n_m = 0$ and $b_1 | n_1, \dots, b_m | n_m$. Following,

$$\begin{aligned} \Lambda^* &:= \{(n_1, \dots, n_m) \in \mathbf{Z}^m : b_1 | n_1, \dots, b_m | n_m\}, \\ \tilde{\Lambda}^* &:= \{(n_1, \dots, n_m) \in \mathbf{Z}^m : n_1 + \dots + n_m = 0, b_1 | n_1, \dots, b_m | n_m\}. \end{aligned}$$

Obviously $\Lambda^* \subset \Lambda$ and $\tilde{\Lambda}^* \subset \tilde{\Lambda}$. From

$$\text{rank} \begin{bmatrix} b_1 b_m & 0 & \dots & 0 & -b_1 b_m \\ 0 & b_2 b_m & \dots & 0 & -b_2 b_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{m-1} b_m & -b_{m-1} b_m \end{bmatrix} = m-1,$$

the set of vectors

$$\{(0, \dots, b_i b_m, \dots, 0, -b_i b_m) : i = 1, \dots, m-1\} \subset \tilde{\Lambda}^*$$

generates the nondegenerated parallelogram of dimension $m-1$, whose volume expressed as a square root of the modulus of the Gram's matrix can be estimated from above by using Hadamard's inequality

$$\sqrt{\det[b_i b_j b_m^2 (1 + \mathbf{1}_{i=j})]_{ij}} \leq 2^{\frac{m-1}{2}} (m-1)^{\frac{m-1}{4}} N^{2\delta(m-1)}.$$

On the other hand, we have also $\tilde{\Lambda}^* \subset \tilde{\Lambda} \simeq \mathbf{Z}^{m-1}$, and then $\tilde{\Lambda}^* \simeq \mathbf{Z}^{m-1}$. We are able to pick such a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subset \tilde{\Lambda}^*$ that it satisfies the assumptions of the Theorem (A.1). The distance between every two points of $\tilde{\Lambda}^*$ is at least $\sqrt{2}$ and we have also the sequence of inequalities

$$\prod_{j=1}^{m-1} \|v_i\| \ll d(\tilde{\Lambda}^*) \ll N^{2\delta(m-1)},$$

hence

$$\|v_j\| \ll N^{2\delta(m-1)}$$

for $j = 1, \dots, m-1$.

Let us consider

$$P := \{t_1 \mathbf{v}_1 + \dots + t_m \mathbf{v}_m : t_1, \dots, t_m \in [0, 1)\}.$$

We will count how many points from lattice $\tilde{\Lambda}$ are contained in every parallelogram of the form $\mathbf{x} + P$ for $\mathbf{x} \in \tilde{\Lambda}^*$. One has

$$\tilde{\Lambda}/\tilde{\Lambda}^* \simeq \Lambda/\Lambda^* \simeq \mathbf{Z}/b_1 \dots b_m \mathbf{Z},$$

thus the answer is $b_1 \dots b_m$.

Let us also define the following subsets

$$\begin{aligned} \tilde{\Lambda}_N &:= \{(n_1, \dots, n_m) \in \mathbf{Z}^m : n_1 + \dots + n_m = N\}, \\ \tilde{\Lambda}_N^* &:= \{(n_1, \dots, n_m) \in \mathbf{Z}^m : n_1 + \dots + n_m = N, b_1 | n_1, \dots, b_m | n_m\}. \end{aligned}$$

If $\mathbf{n} = (n_1, \dots, n_m) \in \mathbf{Z}^m$ is any solution of the equation $b_1 n_1 + \dots + b_m n_m = N$ (which certainly exists because $(b_1, \dots, b_m) = 1$) then we can write $\tilde{\Lambda}_N = \mathbf{n} + \tilde{\Lambda}$ and $\tilde{\Lambda}_N^* = \mathbf{n} + \tilde{\Lambda}^*$. According to this, we can consider the parallelograms of the form $\mathbf{x} + P$ for every $\mathbf{x} \in \tilde{\Lambda}_N^*$. We can say that every one of them contains exactly $b_1 \dots b_m$ points from $\tilde{\Lambda}_N$. Let us define $R := \text{diam } P$. Thus if some ball with radius R contains at least one point from $(\mathbf{x} + P) \cap \tilde{\Lambda}_N^*$, then it contains the whole parallelogram $\mathbf{x} + P$. Note that

$$R \leq \sum_{j=1}^{m-1} \|v_j\| \ll N^{2\delta(m-1)}.$$

The number of points of the form $(n_1, \dots, n_m) \in \mathbf{Z}^m$, which obeys $n_1, \dots, n_m \geq 1$ is equal

$$\binom{N}{m-1} = \frac{N^{m-1}}{(m-1)!} + O(N^{m-2}).$$

For every $\mathbf{x} \in \tilde{\Lambda}_N^*$ the parallelogram $\mathbf{x} + P$ contains exactly one point from $\tilde{\Lambda}_N^*$, therefore $J_{b_1, \dots, b_m}(N)$ is equal to the number of parallelograms of this form contained in the set

$$T := \text{conv}\{(N, 0, \dots, 0), \dots, (0, \dots, 0, N, 0, \dots, 0), \dots, (0, \dots, 0, N)\}$$

with respect to these which have non-empty intersection with ∂T .

Note that $\mathbf{x} \in T \cap \tilde{\Lambda}_N^*$ has the property that $B(\mathbf{x}, R)$ has an empty intersection with ∂T , thus the whole parallelogram $\mathbf{x} + P$ is contained in T . From

$$|\{\mathbf{x} \in T \cap \tilde{\Lambda}_N^* : \text{dist}(\mathbf{x}, \partial T) \leq R\}| \leq |\{\mathbf{x} \in \Lambda : \text{dist}(\mathbf{x}, \partial T) \leq R\}| \ll R^m |\partial T| \ll N^{2\delta(m-1)+m-2},$$

we get that there are at most $O(N^{2\delta(m-1)+m-2})$ parallelograms of the form $\mathbf{x} + P$ for $\mathbf{x} \in T \cap \tilde{\Lambda}_N^*$ which have a non-empty intersection with ∂T . Hence the number of the points which are contained in these parallelograms which are fully contained in T equals $N^{m-1} + O(N^{2\delta(m-1)+m-2})$.

After dividing this value by $b_1 \dots b_m$ we get the number of such parallelograms and the asymptotic behaviour of $J_{b_1, \dots, b_m}(N)$. □

APPENDIX B.

Lemma B.1. *Let $b_1, \dots, b_m \leq N$ be positive integers such that b_1, b_2, b_3 are absolute constants.*

$$R_m(N; b_1, \dots, b_m) := \sum_{p_1 \leq \frac{N}{b_1}, \dots, p_m \leq \frac{N}{b_m}} \mathbf{1}_{b_1 p_1 + \dots + b_m p_m = N} \ll \frac{N^{m-1}}{b_1 \dots b_m \prod_{j=1}^m \log(N/b_j)}.$$

Proof. From Theorem 3.6 we have

$$R_3(N; b_1, b_2, b_3) \ll \frac{N^2}{\log^3 N}$$

for sufficiently large N . We can present $R_m(N; b_1, \dots, b_m)$ in the form

$$\sum_{p_4 \leq \frac{N}{b_4}, \dots, p_m \leq \frac{N}{b_m}} \sum_{p_1 \leq \frac{N}{b_1}, p_2 \leq \frac{N}{b_2}, p_3 \leq \frac{N}{b_3}} \mathbf{1}_{b_1 p_1 + b_2 p_2 + b_3 p_3 = N - b_4 p_4 - \dots - b_m p_m}.$$

The term inside the parentheses is equal to $R_3(N - b_4 p_4 - \dots - b_m p_m; b_1, b_2, b_3)$ so one gets

$$R_m(N; b_1, \dots, b_m) \ll \frac{N^2}{b_1 b_2 b_3 \log^3 N} \sum_{p_4 \leq \frac{N}{b_4}, \dots, p_m \leq \frac{N}{b_m}} 1 \ll$$

$$\frac{N^2}{b_1 b_2 b_3 \log^3 N} \prod_{j=4}^m \frac{N}{b_j \log(N/b_j)} \ll \frac{N^{m-1}}{b_1 \dots b_m \prod_{j=1}^m \log(N/b_j)}. \quad \square$$

Lemma B.2. *Let $x, y \in \mathbf{R}$. Then we have for any $\delta \in (0, 1)$*

$$\sum_{p \leq x^\delta} \frac{1}{p \log \frac{x}{p}} = (1 + o_\delta(1)) \frac{\log \log x}{\log x}.$$

Proof. Use summation by parts and the prime number theorem. □

Theorem B.3 (Landau). *For $k \geq 1$ we have*

$$\sum_{n \leq x} \mathbf{1}_{\Omega(n)=k} = (1 + o_k(1)) \frac{x(\log \log x)^{k-1}}{(k-1)! \log x}.$$

Proof. See [10]. □

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